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Approximation Guarantees for Minimum Rényi Entropy Functional Representations

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Functional Representation Lemma

Given $(X, Y) \sim P_{XY}$, there exists Z $(Z \perp Y)$ and a function $g(\cdot, \cdot)$ such that $X = g(Y, Z)$
i.e.,

$$H(X \mid Y, Z) = 0$$
$$I(Y; Z) = 0$$

Minimum Rényi Entropy Functional Representation

Given : $(X, Y) \sim P_{XY}$

Find : Z (or $P_{Z|XY}$) …..

…with minimum $H_\alpha(Z)$ $(\forall \alpha \geq 0)$

Such that : $Y \perp Z$

$X = g(Y, Z)$

Equivalence to Minimum (Rényi) Entropy Coupling

Given : $(X, Y) \sim P_{XY}$

Find : Z (or $P_{Z|XY}$) …..

…with minimum $H_\alpha(Z)$ $(\forall \alpha \geq 0)$

Such that : $Y \perp Z$

$X = g(Y, Z)$.

Given : $|\mathcal{Y}|$ marginal PMFs $\{P_{X|Y=y}\}_{y\in\mathcal{Y}}$

Find : coupling $C(\{W_y\}_{y\in\mathcal{Y}})$…..

…with minimum $H_\alpha(C)$ $(\forall \alpha \geq 0)$

Such that : $W_y \sim P_{X|Y=y}$; $\forall y \in \mathcal{Y}$.

However …

- Computing $H_\alpha(Z^\star)$ or $H_\alpha(C^\star)$ is a **NP-hard** problem.

- **Lower bounds on** $H_\alpha(Z^\star)$ — Converse type results **[*Shkel-*Yadav '23]**
- **Upper bounds on** $H_\alpha(Z^\star)$ — Achievability type results

*Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations ," in *IEEE International Symposium on Information Theory (ISIT)*, 2023.

Prelude

Let X be a random variable such that $X \sim P_X$:

Information of X :

$$\imath_X(x) := \log\left(\frac{1}{P_X(x)}\right) \;;\text{ w. p. } P_X(x).$$

Information spectrum of X :

$$\mathbb{F}_{\imath_X(t)} = \mathbb{P}[\imath_X(X) \leq t] \quad ; \forall t \in [0,\infty)$$

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Shannon entropy of X :

$$H(X) = \mathbb{E}[\imath_X(X)]$$
$$= \int_0^\infty \left(1 - \mathbb{F}_{\imath_X}(t)\right) dt$$

Rényi entropy of X :

$$H_\alpha(X) = \frac{1}{1-\alpha} \log\left(\mathbb{E}[2^{(1-\alpha)\imath_X(X)}]\right) ;$$
$$\forall \alpha \in [0,\infty)$$

Information-spectrum based Lower Bound

Theorem : Let $(X, Y) \sim P_{XY}$ be supported on countable $\mathcal{X}$ and countable $\mathcal{Y}$. Then, for any $\alpha \in [0,\infty)$ we have

$$H_\alpha(Z^\star) \geq K_\alpha(P_{XY})$$

where, $$K_\alpha(P_{XY}) = \begin{cases} \frac{1}{1-\alpha} \log \left[1 + \int_0^\infty J_\alpha(x)dx\right] & ; \text{if } \alpha \in [0,1) \cup (1,\infty) \\ \int_0^\infty G(x)dx & ; \alpha = 1 \end{cases}$$

such that : $$G(x) := \sup_{y \in \mathcal{Y}} \left(1 - \mathbb{F}_{\iota_{X|Y=y}}(x)\right)$$

$$J_\alpha(x) := (\ln 2)(1-\alpha)G(x)2^{(1-\alpha)x}$$

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This Work …

- Concerned with **Upper Bounds on** $H_\alpha(Z^\star)$ i.e., Achievability type results.

- Approximation analysis based on the Greedy Coupling Algorithm **[Kocaoglu et al. '17]**
  - Let C_Z denote the output of the algorithm
  - $K_\alpha(P_{XY}) \leq H_\alpha(Z^\star)$ - - [from the Lower bound]
  - $K_\alpha(P_{XY}) \leq H_\alpha(Z^\star) \leq H_\alpha(C_Z)$ - - [problem's nature]

  - **Our work :** $H_\alpha(C_Z) \leq K_\alpha(P_{XY}) + Q$; (finding the smallest Q for every $\alpha \in [0,\infty)$).

Murat Kocaoglu, Alexandros G. Dimakis, Sriram Vishwanath, and Babak Hassibi, "Entropic causal inference", ***In Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence (AAAI'17),*** **AAAI Press, 1156–1162.**

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  - **Our work :** $H_\alpha(C_Z) \leq K_\alpha(P_{XY}) + Q$; (finding the smallest Q for every $\alpha \in [0,\infty)$).
  - For the rest of the presentation : $m := |\mathscr{Y}|$ and $n := |\mathscr{X}|$

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Greedy Coupling Algorithm

- **Input :** m PMFs $\{P_{X|Y=y_i}\}_{i=1}^{m}$, each with $\leq n$ states
- **Output :** Coupling $C_Z := (c_1, c_2, \ldots, c_T)$

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- **Output :** Coupling $C_Z := (c_1, c_2, \ldots, c_T)$
  - Sort each PMF in the non-increasing order
  - Find the minimum of maximum of each PMF i.e., $q = \min_i(P_{X|Y=y_i}(1))$
  - Append q as the next state of C_Z
  - Update the maximum state of each PMF

    i.e., $P_{X|Y=y_i}(1) = \left(P_{X|Y=y_i}(1) - q\right), \ \forall\, i \leq m$
  - Sort each PMF in non-increasing order
  - Find $q = \min_i(P_{X|Y=y_i}(1))$

Repeat until $q = 0$ (applies to the last four steps)

Greedy Coupling Algorithm : Example

- **Input :** $\{P_{X|Y=y_1} = (0.5,0.4,0.1)\,;\, P_{X|Y=y_2} = (0.6,0.2,0.2)\}$; $(m = 2\,,\, n = 3)$

Iteration (t)	Current PMFs	q	Updated PMFs	C_Z
1	(0.5, 0.4, 0.1) (0.6, 0.2, 0.2)	0.5	(0, 0.4, 0.1) (0.1, 0.2, 0.2)	(0.5)
2	(0.4, 0.1, 0) (0.2, 0.2, 0.1)	0.2	(0.2, 0.1, 0) (0, 0.2, 0.1)	(0.5, 0.2)
3	(0.2, 0.1, 0) (0.2, 0.1, 0)	0.2	(0, 0.1, 0) (0, 0.1, 0)	(0.5, 0.2, 0.2)
T = 4	(0.1, 0, 0) (0.1, 0, 0)	0.1	(0, 0, 0) (0, 0, 0)	(0.5, 0.2, 0.2, 0.1)
5	(0, 0, 0) (0, 0, 0)	0	(0, 0, 0) (0, 0, 0)	

- **Output :** Coupling $C_Z = (0.5,0.2,0.2,0.1)$

Connecting the dots…

- Recall, our goal : $H_\alpha(C_Z) \leq K_\alpha(P_{XY}) + Q$
- Also, recall that $K_\alpha(P_{XY})$ is a function of $G(x)$.

$$G(x) := \sup_{y \in \mathcal{Y}} \left(1 - \mathbb{F}_{\iota_{X|Y=y}}(x)\right)$$

$$J_\alpha(x) := (\ln 2)(1-\alpha)G(x)2^{(1-\alpha)x}$$

$$K_\alpha(P_{XY}) = \begin{cases} \frac{1}{1-\alpha} \log \left[1 + \int_0^\infty J_\alpha(x)dx\right] & ; \text{if } \alpha \in [0,1) \cup (1,\infty) \\ \int_0^\infty G(x)dx & ; \alpha = 1 \end{cases}$$

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- Track the behavior of $G(x)$ at every iteration of the greedy algorithm.

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$$G^{t+1}(x) \begin{cases} = G^t(x) - p_1^t(1) & ; x < a_1 \\ \leq G^t(x) + (p_m^t(1) - p_1^t(1)) & ; x \in [a_1, a_2) \\ \vdots & \\ \leq G^t(x) + (p_2^t(1) - p_1^t(1)) & ; x \in [a_{m-1}, a_m) \\ = G^t(x) & ; x \geq a_m \end{cases}$$

Where,

$$P_i := P_{X|Y=y_i} \quad \forall i \leq m$$

$$a_1 = \log \frac{1}{p_1^t(1)}$$

$$a_2 = \log \frac{1}{p_m^t(1) - p_1^t(1)}$$

$$\vdots$$

$$a_m = \log \frac{1}{p_2^t(1) - p_1^t(1)}$$

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- Also, recall that $K_\alpha(P_{XY})$ is a function of $G(x)$.

- Track the behavior of $G(x)$ at every iteration of the greedy algorithm.
- $J_\alpha(x)$ is a function of $G(x)$ i.e., $J_\alpha(x) = h(\alpha, x)G(x)$ where $h(\alpha, x) = \ln 2(1-\alpha)2^{(1-\alpha)x}$

$$J_\alpha^{t+1}(x) \begin{cases} = J_\alpha^t(x) - h(\alpha, x)p_1^t(1) & ; x < a_1 \\ \leq J_\alpha^t(x) + h(\alpha, x)(p_m^t(1) - p_1^t(1)) & ; x \in \left[a_1, a_2\right) \\ \vdots & \\ \leq J_\alpha^t(x) + h(\alpha, x)(p_2^t(1) - p_1^t(1)) & ; x \in \left[a_{m-1}, a_m\right) \\ = J_\alpha^t(x) & ; x \geq a_m \end{cases}$$

Connecting the dots…

- Track the behavior of $G(x)$ at every iteration of the greedy algorithm.
- $J_\alpha(x)$ is a function of $G(x)$ i.e., $J_\alpha(x) = h(\alpha, x)G(x)$ where $h(\alpha, x) = \ln 2(1-\alpha)2^{(1-\alpha)x}$
- Consequently,

$$\int_0^\infty J_\alpha^{t+1}(x)dx - \int_0^\infty J_\alpha^t(x)dx \le p_1^t(1) - (p_1^t(1))^\alpha\left[1 - \tilde{r}(\alpha, m)\right]$$

where, $\tilde{r}(\alpha, m) := \begin{cases} \max_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}.}} \sum_{k=2}^m w_k(w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); \text{ for } \alpha \in [0,1), \\ \min_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}.}} \sum_{k=2}^m w_k(w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); \text{ for } \alpha \in (1,\infty). \end{cases}$; and $w_k := \dfrac{p_k^t(1) - p_1^t(1)}{p_1^t(1)}$.

- Sum over all iterations of the greedy algorithm, $1 \le t \le T$.

Connecting the dots…

- Consequently,

$$\int_0^\infty J_\alpha^{t+1}(x)dx - \int_0^\infty J_\alpha^t(x)dx \leq p_1^t(1) - (p_1^t(1))^\alpha\left[1 - \tilde{r}(\alpha, m)\right]$$

- Sum over all iterations of the greedy algorithm, $1 \leq t \leq T$.

$$1 + \int_0^\infty J_\alpha^1(x)dx \geq \left[r(\alpha, m))\right] \sum_{t=1}^{T} (p_1^t(1))^\alpha$$

Where $r(\alpha, m) := \max(0, 1 - \tilde{r}(\alpha, m)$.

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Where $r(\alpha, m) := \max(0, 1 - \tilde{r}(\alpha, m))$.

- On taking logarithm on both sides,

$$\frac{1}{1-\alpha} \log\left(1 + \int_0^\infty J_\alpha^1(x)dx\right) \geq \frac{1}{1-\alpha} \log\left[r(\alpha, m)\right] + \frac{1}{1-\alpha} \log\left(\sum_{t=1}^{T} (p_1^t(1))^\alpha\right)$$

Connecting the dots…

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$K_\alpha(P_{XY})$

$H_\alpha(C_Z)$

Main Results

Theorem : Let $(X, Y) \sim P_{XY}$ be supported on countable $\mathcal{X}$ and countable $\mathcal{Y}$. Then, for any $\alpha \in [0,\infty)$ we have

$$H_\alpha(C_Z) \leq K_\alpha(P_{XY}) + F(\alpha, m)$$

where, $$F(\alpha, m) = \frac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$$
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$$r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m)).$$

Consequently,

$$K_\alpha(P_{XY}) \leq H^\star_\alpha(Z) \leq H_\alpha(C_Z) \leq K_\alpha(P_{XY}) + F(\alpha, m)$$
$$\leq H^\star_\alpha(Z) + F(\alpha, m)$$

Main Results

Corollary 1 : Let $(X, Y) \sim P_{XY}$ be supported on countable $\mathcal{X}$ and binary $\mathcal{Y}$ (i.e., $m = 2$). Then, for any $\alpha \in [0,\infty)$, we have

$$H_\alpha(C_Z) \le K_\alpha(P_{XY}) + F(\alpha,2)$$

where, $$F(\alpha,2) = \frac{1}{\alpha - 1} \log \left[1 + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \right].$$

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$F(\alpha, m)$ doesnot have a closed-form solution, in general!

Main Results

Recall that $F(\alpha, m) = \dfrac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$; where $r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m))$ such that

$$\tilde{r}(\alpha, m) := \begin{cases} \max_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}.}} \sum_{k=2}^{m} w_k (w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); & \text{for } \alpha \in [0,1), \\ \min_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}.}} \sum_{k=2}^{m} w_k (w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); & \text{for } \alpha \in (1,\infty). \end{cases}$$

Lemma : For every $\alpha \in [0,\infty)$, $F(\alpha, m)$ is an non-decreasing function of m.

As $m \to \infty$, $F(\alpha, m)$ approaches $\dfrac{1}{\alpha - 1} \log \left[\max \left(0, \dfrac{2\alpha - 1}{\alpha} \right) \right]$.

Main Results

Corollary 2 : Let $(X, Y) \sim P_{XY}$ be supported on countable $\mathcal{X}$ and countable $\mathcal{Y}$. Then, for any $\alpha \in [0,\infty)$, we have

$$H_\alpha(C_Z) \leq K_\alpha(P_{XY}) + \lim_{m\to\infty} F(\alpha, m)$$
$$= K_\alpha(P_{XY}) + \frac{1}{\alpha - 1} \log\left[\max\left(0, \frac{2\alpha - 1}{\alpha}\right)\right].$$

Comparison of Upper Bounds : ($m = 2$)

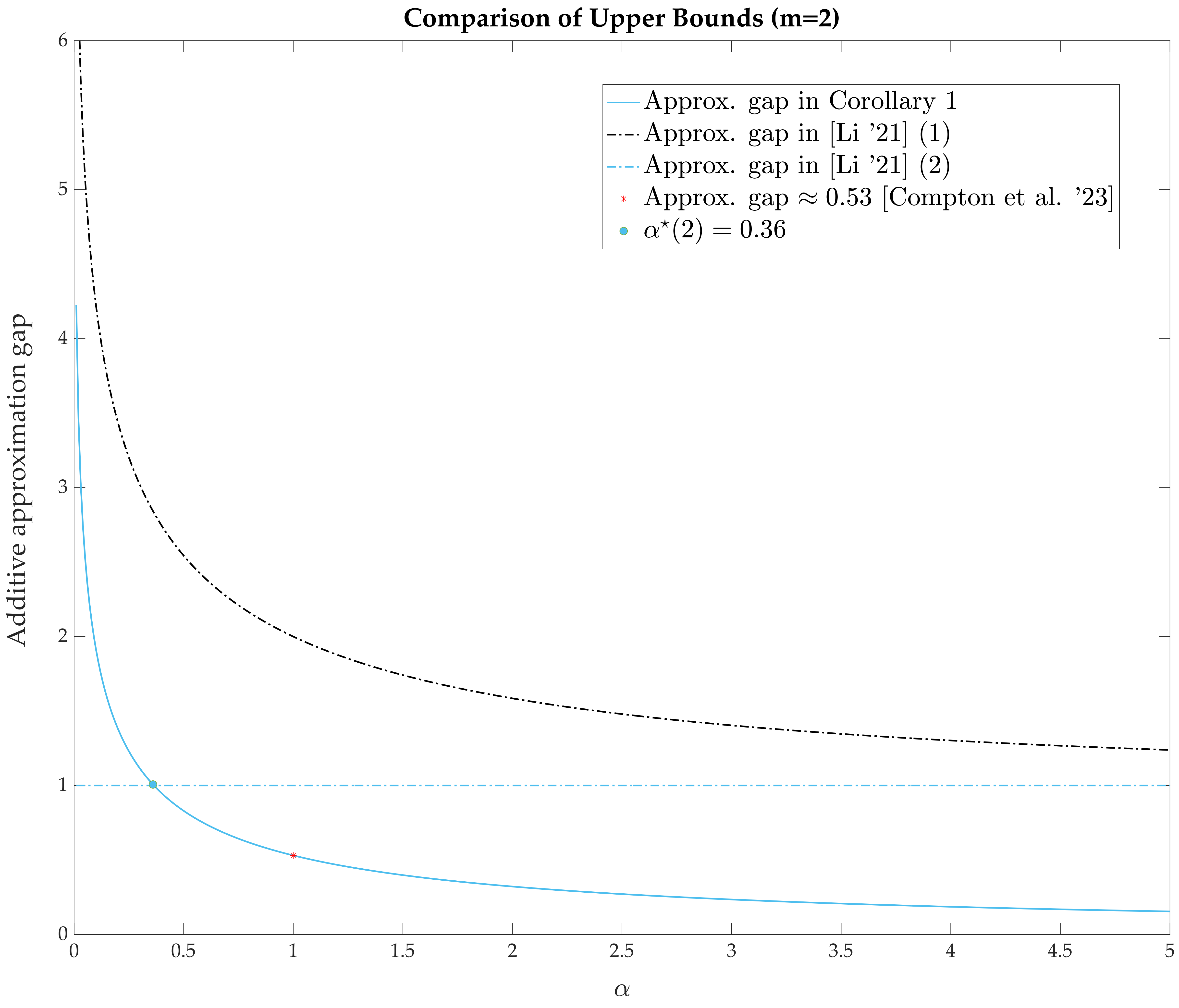
[Li, Trans. IT ’21] (1) : $H_\alpha(\tilde{Z}) \leq H_\alpha(Z^\star) + \begin{cases} \infty & ; \text{ if } \alpha = 0 \\ 2 & ; \text{ if } \alpha = 1 \\ 1 & ; \text{ if } \alpha = \infty \\ \frac{-\alpha - \log(1 - 2^{-\alpha})}{1 - \alpha} & ; \text{ otherwise} \end{cases}$

[Li, Trans. IT ’21] (2) : $H_\alpha(\tilde{Z}) \leq H_\alpha(Z^\star) + 1.$

[Compton et al., AISTATS ’23] : $H_1(C_Z) \leq H_1(Z^\star) + \frac{\log_2 e}{e} \approx 0.53$ **. (Only for Shannon Entropy)**

[Our Work] : $H_\alpha(C_Z) \leq H_\alpha(Z^\star) + F(\alpha,2).$

Comparison of Upper Bounds : ($m = 2$)



Comparison of Upper Bounds : (arbitrary m)

[Li, Trans. IT '21] (2) : $H_\alpha(\tilde{Z}) \leq H_\alpha(Z^\star) + \frac{1}{1-\alpha} \log \left(\frac{(2^\alpha - 2)2^{-\alpha m} + 2^{-\alpha}}{1 - 2^{-\alpha}} \right).$

[Compton, ISIT '22] : $H_1(C_Z) \leq H_1(Z^\star) + \log_2 e \approx 1.44$ **. (Only for Shannon Entropy)**

[Compton et al., AISTATS '23] : $H_1(C_Z) \leq H_1(Z^\star) + \frac{1 + \log_2 e}{e} \approx 1.22$ **. (Only for Shannon Entropy)**

[Our Work] : $H_\alpha(C_Z) \leq H_\alpha(Z^\star) + F(\alpha, m) \leq H_\alpha(Z^\star) + \frac{1}{\alpha - 1} \log \left[\max \left(0, \frac{2\alpha - 1}{\alpha} \right) \right].$

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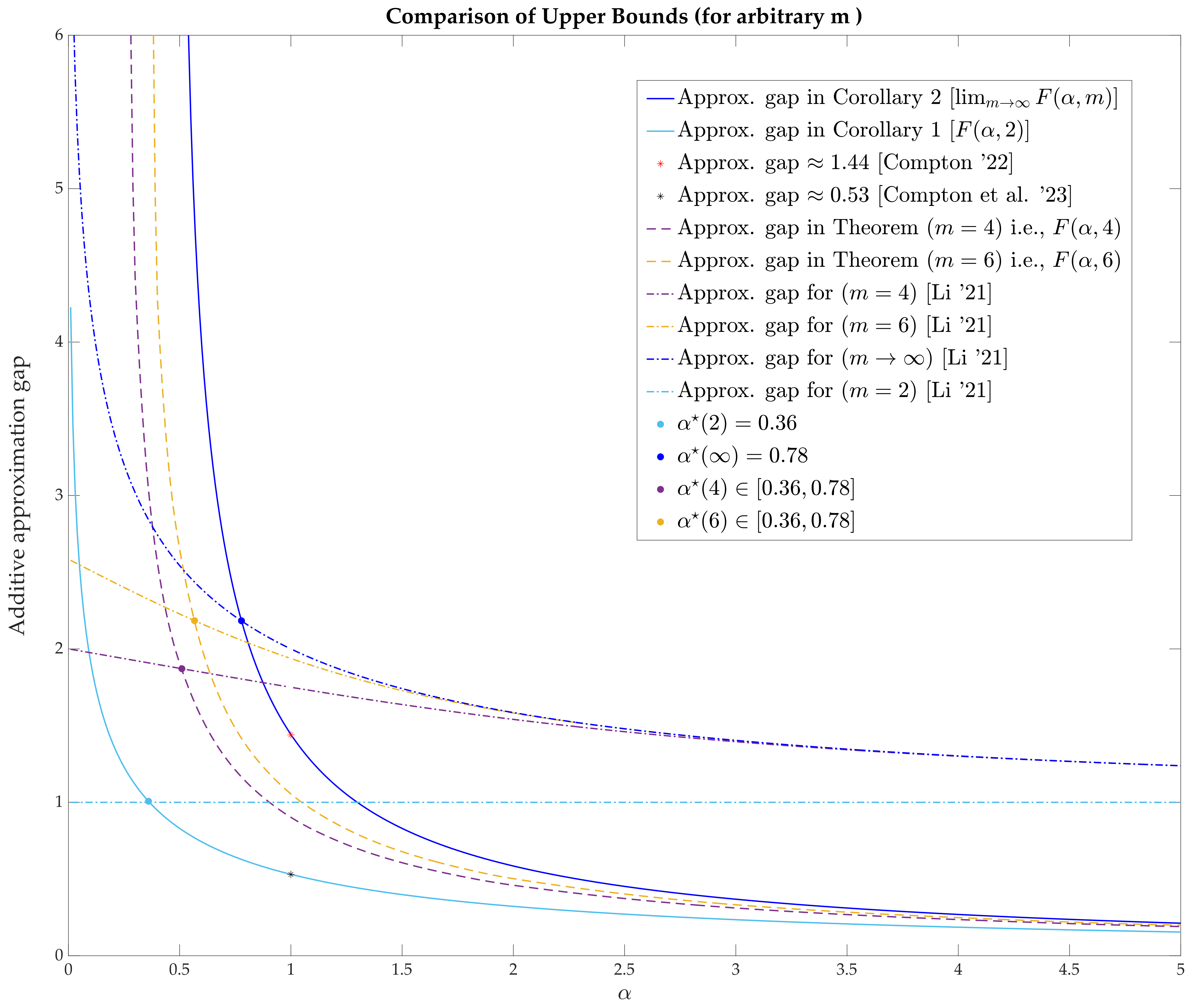

Summary

- Achievability type results **(Upper Bounds)** for Minimum Rényi Entropy Couplings and Functional Representations.
  - Approximation Analysis between the Rényi entropy of the 'output of the Greedy Coupling Algorithm' and the 'optimal coupling' i.e.,

$$H_\alpha(C_Z) \leq H_\alpha^\star(Z) + F(\alpha, m)$$

  - Our analysis is better for high values of α i.e., $\alpha \geq a^\star(m)$, where $\alpha^\star(m) \in [0.36,0.78]$ for every $m \geq 2$.
  - Greedy Coupling Algorithm is optimal for min-entropy i.e., $\alpha \to \infty$.