Minimum Rényi Entropy Couplings and their Applications

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(Joint work with Yanina Shkel)

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Outline

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Minimum Rényi Entropy Coupling (M-REC)

Given : m marginal PMFs $\{P_i\}_{i=1}^m$

Find : coupling $C^\star \in \underset{C \in \Gamma(P_1,\ldots,P_m)}{\arg\min} H_\alpha(C), \qquad \alpha \geq 0$

where $\Gamma(P_1, \ldots, P_m)$ is the set of all couplings with marginals $\{P_i\}_{i=1}^m$.

However . . .

- Let $C^\star$ denote the optimal coupling.
- Computing $C^\star$ (or $H_\alpha(C^\star)$) is **NP-hard** in the support size of PMFs.

M. Kovačević et al., "On the entropy of couplings", Information and Computation, 2015.

However ...

- Let $C^\star$ denote the optimal coupling.
- Computing $C^\star$ (or $H_\alpha(C^\star)$) is **NP-hard** in the support size of PMFs.

- **Lower bounds on** $H_\alpha(C^\star)$ — Converse type results
- **Upper bounds on** $H_\alpha(C^\star)$ — Achievability type results

M. Kovačević et al., "On the entropy of couplings", Information and Computation, 2015.

Outline

Lower bounds (Converse Results)

What are the worst-case lower bounds on $H_\alpha(C^\star)$?

Prelude

Let P be a PMF such that $P = (p_1, p_2, p_3 \ldots,)$.

Information of P :

$$\imath_P = \log\left(\frac{1}{p_i}\right);$$

with probability p_i.

Information spectrum of P :

$$\mathbb{F}_{\imath_P}(t) = \mathbb{P}[\imath_P \leq t] ;$$

$$\forall t \in [0, \infty).$$

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$$\forall t \in [0, \infty).$$

Shannon entropy of P :

$$H(P) = \mathbb{E}[\imath_P]$$

$$= \int_0^\infty \left(1 - \mathbb{F}_{\imath_P}(t)\right) dt.$$

Rényi entropy of P :

$$H_\alpha(P) = \frac{1}{1-\alpha} \log\left(\mathbb{E}\left[2^{(1-\alpha)\imath_P}\right]\right) ;$$

$$\forall \alpha \in [0, \infty).$$

Majorization ($\preceq_m$)

Definition

Given PMFs

$$Q = (q_1, q_2, q_3, \dots,) \ ; \quad q_1 \geq q_2 \geq \dots$$
$$P = (p_1, p_2, p_3, \dots,) \ ; \quad p_1 \geq p_2 \geq \dots$$

we say $\quad Q \preceq_m P$

if $$\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i, \quad \forall k \geq 1.$$

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• $\preceq_m$ is a partial order on the set of all ordered PMFs. Moreover, it forms a complete lattice.

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Greatest Lower Bound of $\{P_1, P_2, \dots, P_m\}$

Let $\mathcal{F} = \{Q \colon Q \preceq_m P_i \quad \forall i \in [m]\}$

$$\exists \bigwedge_{i=1}^{m} P_i \in \mathcal{F} \ \text{ s.t.}$$

$$Q \preceq_m \bigwedge_{i=1}^{m} P_i \preceq_m P_i \ ; \forall i, \ Q.$$

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Schur Concavity

$$Q \preceq_m P \implies f(Q) \geq f(P)$$

Majorization ($\preceq_m$)

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Schur Concavity

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Rényi entropy is Schur concave.

Lower bound : A very basic one

- $C(P_1, \cdots, P_m) \sqsubseteq P_i \;\; ; \; \forall C, \; \forall i \in [m]$
- Aggregation implies Majorization.
- $C(P_1, \cdots, P_m) \preceq_m P_i \;\; ; \; \forall C, \; \forall i \in [m]$

$$\boxed{H_\alpha(C^\star) \geq \max_{i \in [m]} H_\alpha(P_i)}$$

Y. Y. Shkel, R. Blum, H. V. Poor, "Secrecy by design with applications to privacy and compression," IEEE Transactions on Information Theory, vol. 67, no. 2, pp. 824-843, 2021.

Lower bound : A very basic one

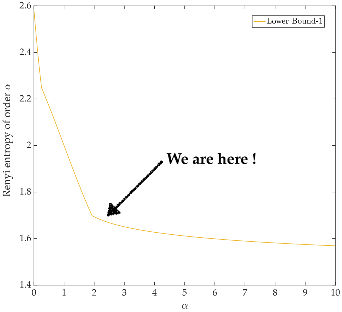
$$\boxed{H_\alpha(C^\star) \geq \max_{i \in [m]} H_\alpha(P_i)}$$

Toy Example with $(m = 3,\ n = 6)$:

$P_1 = (0.5, 0.125, 0.125, 0.125, 0.125, 0)$;

$P_2 = (0.4,\ 0.4,\ 0.1,\ 0.1,\ 0,\ 0)$;

$P_3 = (0.35, 0.35, 0.25, 0.04, 0.005, 0.005)$;



Lower bound : Based on Majorization ($\preceq_m$)

Recall,

$$C(P_1, \cdots, P_m) \preceq_m P_i \;\; ; \forall C, \, \forall i \in [m]$$

Thus,

$$C(P_1, \cdots, P_m) \;\preceq_m\; \bigwedge_{i=1}^{m} P_i \;\preceq_m\; P_i \;\; ; \quad \forall i \in [m]$$

$$\boxed{H_\alpha(C^\star) \geq H_\alpha\Big(\bigwedge_{i=1}^{m} P_i\Big)}$$

F. Cicalese, L. Gargano, and U. Vaccaro, "Minimum-entropy couplings and their applications," IEEE Transactions on Information Theory, vol. 65, no. 6, pp. 3436–3451, 2019.

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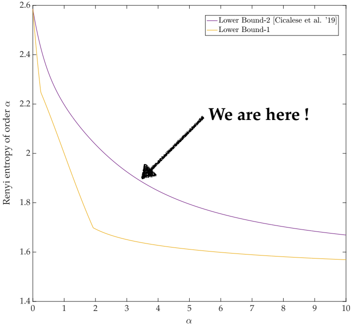

F. Cicalese, L. Gargano, and U. Vaccaro, "Minimum-entropy couplings and their applications," IEEE Transactions on Information Theory, vol. 65, no. 6, pp. 3436–3451, 2019.

Majorization : Information-spectrum sense ($\preceq_\imath$)

We say : P majorizes Q in an information-spectrum sense, i.e.,

$$Q \preceq_\imath P$$

if $$\boxed{\mathbb{F}_{\imath_Q}(t) \leq \mathbb{F}_{\imath_P}(t), \quad \forall t \in [0, \infty)}$$

*Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations," in IEEE International Symposium on Information Theory (ISIT), 2023.

Information-spectrum based Lower Bound : Main Result 1

Theorem: Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions with support size at most n. Then,

$$C \preceq_{\imath} P_i \quad ; \quad \forall C, \; \forall i \in [m]$$

or

$$\mathbb{F}_{\imath_C}(t) \leq \mathbb{F}_{\imath_{P_i}}(t) \; ; \quad \forall t \in [0, \infty) \, , \forall C, \; \forall i \in [m]$$

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Information-spectrum based Lower Bound : Main Result 1

$$\mathbb{F}_{\imath_{C^\star}}(t) \le \mathbb{F}_{\imath_{P_i}}(t) \; ; \quad \forall i \in [m]$$

$$\Downarrow$$

$$\begin{aligned} H(C^\star) = \mathbb{E}[\imath_{C^\star}(C^\star)] &= \int_0^\infty \Big(1 - \mathbb{F}_{\imath_{C^\star}}(t)\Big) dt \\ &\ge \int_0^\infty \max_{i\in[m]} \Big(1 - \mathbb{F}_{\imath_{P_i}}(t)\Big) dt \end{aligned}$$

$$\boxed{H(C^\star) \ge K_1(\mathcal{S})}$$

where $K_1(\mathcal{S}) := \int_0^\infty \max_{i\in[m]} \Big(1 - \mathbb{F}_{\imath_{P_i}}(t)\Big) dt$

Similar extension for Rényi Entropy
i.e., $H_\alpha(C^\star) \ge K_\alpha(\mathcal{S})$.

Information-spectrum based Lower Bound : Main Result 1

$$\mathbb{F}_{\imath_{C^\star}}(t) \leq \mathbb{F}_{\imath_{P_i}}(t) \; ; \quad \forall i \in [m]$$

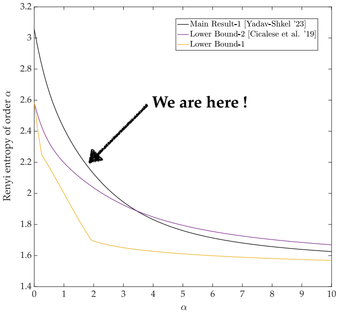
$$\Downarrow$$

$$H(C^\star) = \mathbb{E}[\imath_{C^\star}(C^\star)] = \int_0^\infty \Big(1 - \mathbb{F}_{\imath_{C^\star}}(t)\Big) dt$$

$$\geq \int_0^\infty \max_{i\in[m]} \Big(1 - \mathbb{F}_{\imath_{P_i}}(t)\Big) dt$$

$$\boxed{H(C^\star) \geq K_1(\mathcal{S})}$$

where $K_1(\mathcal{S}) := \int_0^\infty \max_{i\in[m]} \Big(1 - \mathbb{F}_{\imath_{P_i}}(t)\Big) dt$



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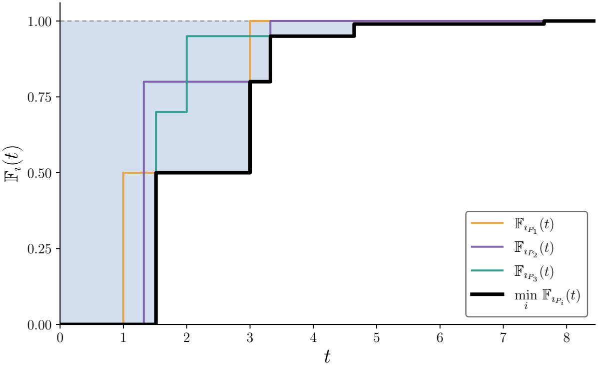
i.e., $H_\alpha(C^\star) \geq K_\alpha(\mathcal{S})$.

Information-spectrum based Lower bound: Main Result 1

Example: Shannon entropy

- $H(C^\star) \geq K_1(\mathcal{S}) =$ area above the lower envelope.



Information-spectrum based Lower Bound : Main Result 1

Theorem: Let $\mathcal{S} := \{P_1, \dots, P_m\}$ be the set of m marginal distributions. Then, for any $\alpha \in [0, \infty)$, we have

$$H_\alpha(C^\star) \geq K_\alpha(\mathcal{S})$$

where,

$$K_\alpha(\mathcal{S}) = \begin{cases} \dfrac{1}{1-\alpha} \log \left[1 + \displaystyle\int_0^\infty J_\alpha(t)\, dt\right] & ; \text{ if } \alpha \in [0,1) \cup (1, \infty) \\ \displaystyle\int_0^\infty G(t)\, dt & ; \ \alpha = 1 \end{cases}$$

such that :

$$G(t) := \max_{i \in [m]} \left(1 - \mathbb{F}_{\imath_{P_i}}(t)\right)$$

$$J_\alpha(t) := (\ln 2)(1-\alpha)\, G(t)\, 2^{(1-\alpha)t}$$

*Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations," in IEEE International Symposium on Information Theory (ISIT), 2023.

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We say :

$$Q \preceq_\imath P$$
$$\text{if} \quad \mathbb{F}_{\imath_Q}(t) \leq \mathbb{F}_{\imath_P}(t), \quad \forall t \in [0, \infty)$$

Recall that :

$$Q \preceq_m P$$
$$\text{if} \quad \sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i \quad \forall k \geq 1$$

- **Lemma 1:** $Q \preceq_\imath P \implies Q \preceq_m P$

Majorization : Information-spectrum sense ($\preceq_\imath$)

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- **Lemma 1:** $Q \preceq_\imath P \implies Q \preceq_m P$
- Let $\mathcal{F} = \{Q \colon Q \preceq_\imath P_i \; ; \; \forall i \in [m]\}$

$$\nexists \bigwedge_{i=1}^{m} P_i \in \mathcal{F} \quad \text{s.t.} \quad Q \preceq_\imath \bigwedge_{i=1}^{m} P_i \preceq_\imath P_i \; ; \; \forall i \in [m] \, , \; Q \in \mathcal{F}$$

- $\preceq_\imath$ does not form a lattice; the greatest lower bound does not exist.

Majorization : Information-spectrum sense ($\preceq_\imath$)

We say :

$$Q \preceq_\imath P$$
if $\mathbb{F}_{\imath_Q}(t) \leq \mathbb{F}_{\imath_P}(t), \quad \forall t \in [0, \infty)$

Recall that :

$$Q \preceq_m P$$
if $\displaystyle\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i \quad \forall k \in [m]$

- **Lemma 1:** $Q \preceq_\imath P \implies Q \preceq_m P$
- **Lemma 2:** Let $\mathcal{F} = \{Q\colon Q \preceq_\imath P_i \quad \forall i \in [m]\}$

$$\exists Q^* \in \mathcal{F} \text{ s.t. } Q \preceq_m Q^* \preceq_i P_i\ ; \quad \forall Q \in \mathcal{F},\, i \in [m].$$

- Q^* is computable in polynomial time $\mathcal{O}(mn)$.

*Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations," in IEEE International Symposium on Information Theory (ISIT), 2023.

Information-spectrum based Lower bound: Main Result II

Recall : $C \preceq_\imath P_i$; $\forall C,\ \forall i \in [m]$

Define : $\mathcal{F} = \{Q\colon Q \preceq_\imath P_i\ ;\ i \in [m]\}$

$\forall C$, we have that $C \in \mathcal{F}$. From Lemma 2,

$$\exists Q^* \in \mathcal{F} \text{ s.t. } Q \preceq_m Q^* \preceq_\imath P_i \ ;$$

Therefore,

$$C^\star \preceq_m Q^* \preceq_\imath P_i \ ;\ \forall i \in [m].$$

$$\boxed{H_\alpha(C^\star) \geq H_\alpha(Q^*)}$$

*Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations," in IEEE International Symposium on Information Theory (ISIT), 2023.

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$$\exists Q^* \in \mathcal{F} \;\; \text{s.t.} \;\; Q \preceq_m Q^* \preceq_\imath P_i \;\; ;$$

Therefore,

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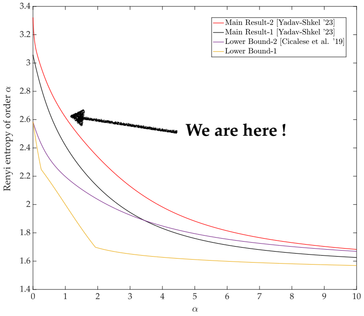
$$\boxed{H_\alpha(C^\star) \geq H_\alpha(Q^*)}$$

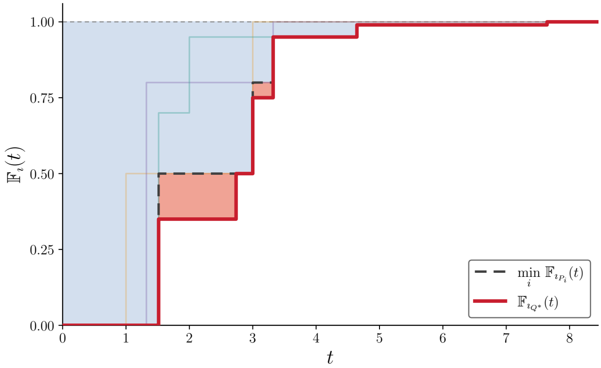


[*] Y. Y. Shkel, and [*] A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations," in IEEE International Symposium on Information Theory (ISIT), 2023.

Information-spectrum based Lower bound: Main Result 2

Example: Shannon entropy

- Consider $\mathcal{F} = \{Q : Q \preceq_\imath P_i \; ; \; \forall i \in [m]\}$
- Q^*: the $\preceq_m$-greatest element of $\mathcal{F}$ [Lemma 2]
- $H(C^\star) \geq H(Q^*) \geq K_1(\mathcal{S})$: Q^* recovers strictly more area.



Outline

Upper bounds (Achievability Results)

Can we construct 'nearly-optimal' couplings and give approximation guarantees w.r.t $H_\alpha(C^\star)$?

Approximation Algorithms

- Greedy Coupling Algorithm [Painsky et al.]
  - Let C_G denote the output of the algorithm.
- Comonotone Coupling Algorithm [Yadav-Shkel]
  - Let C_M denote the output of the algorithm.

A. Painsky, S. Rosset, and M. Feder, "Memoryless representation of Markov processes", in IEEE International Symposium on Information Theory (ISIT), 2013.

A. K. Yadav, and Y. Y. Shkel, "Couplings and Entropic Inequalities on the Majorization Lattice," 2026. (In part, accepted at IEEE Information Theory Workshop (ITW), 2026.)

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- $H_\alpha(\bigwedge \mathcal{S})/K_\alpha(\mathcal{S}) \leq H_\alpha(Q^*) \leq H_\alpha(C^\star)$ - - [from the Lower bound]
- $H_\alpha(C^\star) \leq H_\alpha(C_G)/H_\alpha(C_M)$ - - [problem's nature]

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- $H_\alpha(\bigwedge \mathcal{S})/K_\alpha(\mathcal{S}) \leq H_\alpha(Q^*) \leq H_\alpha(C^\star)$ - - [from the Lower bound]
- $H_\alpha(C^\star) \leq H_\alpha(C_G)/H_\alpha(C_M)$ - - [problem's nature]
- **Our Goal :** $H_\alpha(C_G)/H_\alpha(C_M) \leq H_\alpha(C^\star) + Q$; (smallest Q for every $\alpha \in [0, \infty]$).

Greedy Coupling Algorithm

- **Input :** m PMFs $\{P_i\}_{i=1}^m$, each with $\leq n$ states
- **Output :** Coupling $C_G := (c_1, c_2, \ldots, c_T)$
- **Algorithm :**
  - Sort each PMF in the non-increasing order
  - Find the minimum of maximum of each PMF i.e., $q = \min_i(P_i(1))$

$$\left.\begin{array}{l} \circledast \text{ Append } q \text{ as the next state of } C_G \\ \circledast \text{ Update the maximum state of each PMF} \\ \quad \text{i.e., } P_i(1) = \big(P_i(1) - q\big)\,,\ \forall\, i \leq m \\ \circledast \text{ Sort each PMF in non-increasing order} \\ \circledast \text{ Find } q = \min_i(P_i(1)) \end{array}\right\} \quad \begin{array}{c} \textbf{Repeat} \\ \textbf{until} \\ \boldsymbol{q > 0} \end{array}$$

Greedy Coupling Algorithm : Example

- **Input :** $\{P_1 = (0.5, 0.4, 0.1);\ P_2 = (0.6, 0.2, 0.2)\};\ \ (m = 2\,,\ n = 3)$

Iteration (t)	**Current PMFs**	**q**	**Updated PMFs**	C_G
1	(0.5, 0.4, 0.1) (0.6, 0.2, 0.2)	0.5	(0, 0.4, 0.1) (0.1, 0.2, 0.2)	(0.5)
2	(0.4, 0.1, 0) (0.2, 0.2, 0.1)	0.2	(0.2, 0.1, 0) (0, 0.2, 0.1)	(0.5, 0.2)
3	(0.2, 0.1, 0) (0.2, 0.1, 0)	0.2	(0, 0.1, 0) (0, 0.1, 0)	(0.5, 0.2, 0.2)
$\mathrm{T} = 4$	(0.1, 0, 0) (0.1, 0, 0)	0.1	(0, 0, 0) (0, 0, 0)	(0.5, 0.2, 0.2, 0.1)
5	(0, 0, 0) (0, 0, 0)	0	(0, 0, 0) (0, 0, 0)	

- **Output :** Coupling $C_G = (0.5, 0.2, 0.2, 0.1)$

Main Results

Theorem: Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions of support size n. Then, for any $\alpha \in [0, \infty)$, we have

$$H_\alpha(C_G) \leq K_\alpha(\mathcal{S}) + F(\alpha, m)$$

where,

$$F(\alpha, m) = \frac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$$
$$r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m)).$$

where, $$\tilde{r}(\alpha, m) := \begin{cases} \displaystyle\max_{\substack{w_1=0;\ w_{m+1}=1; \\ w_1<w_2\leq w_3\leq\ldots\leq w_m<w_{m+1}}} \sum_{k=2}^{m} w_k \left(w_k^{\alpha-1} - w_{k+1}^{\alpha-1}\right); & \text{for } \alpha \in [0,1), \\ \displaystyle\min_{\substack{w_1=0;\ w_{m+1}=1; \\ w_1<w_2\leq w_3\leq\ldots\leq w_m<w_{m+1}}} \sum_{k=2}^{m} w_k \left(w_k^{\alpha-1} - w_{k+1}^{\alpha-1}\right); & \text{for } \alpha \in (1,\infty). \end{cases}$$

A. K. Yadav, and Y. Y. Shkel, "Approximation Guarantees for Minimum Rényi Entropy Functional Representations", in IEEE International Symposium on Information Theory (ISIT), 2025.

Main Results

Corollary 1 : Let $\mathcal{S} := \{P_1, P_2\}$ be the set of two marginal distributions of support size n. Then, for any $\alpha \in [0, \infty)$, we have

$$H_\alpha(C_G) \leq K_\alpha(\mathcal{S}) + F(\alpha, 2)$$

where, $$F(\alpha, 2) = \frac{1}{\alpha - 1} \log \left[1 + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \right].$$

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$$H_\alpha(C_G) \leq K_\alpha(\mathcal{S}) + F(\alpha, 2)$$

where, $$F(\alpha, 2) = \frac{1}{\alpha - 1} \log \left[1 + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \right].$$

$F(\alpha, m)$ does not have a closed-form solution, in general!

A. K. Yadav, and Y. Y. Shkel, "Approximation Guarantees for Minimum Rényi Entropy Functional Representations", in IEEE International Symposium on Information Theory (ISIT), 2025.

Main Results

Recall that $F(\alpha, m) = \dfrac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$; where
$r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m))$ such that

$$\tilde{r}(\alpha, m) := \begin{cases} \max\limits_{\substack{w_1=0;\ w_{m+1}=1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}}} \sum_{k=2}^{m} w_k \left(w_k^{\alpha-1} - w_{k+1}^{\alpha-1} \right); & \text{for } \alpha \in [0, 1), \\ \min\limits_{\substack{w_1=0;\ w_{m+1}=1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}}} \sum_{k=2}^{m} w_k \left(w_k^{\alpha-1} - w_{k+1}^{\alpha-1} \right); & \text{for } \alpha \in (1, \infty). \end{cases}$$

Lemma: For every $\alpha \in [0, \infty)$, $F(\alpha, m)$ is a non-decreasing function of m.

As $m \to \infty$, $F(\alpha, m)$ approaches $\dfrac{1}{\alpha - 1} \log \left[\max \left(0, \dfrac{2\alpha - 1}{\alpha} \right) \right]$.

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Main Results

Corollary 2 : Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions of support size n. Then, for any $\alpha \in [0, \infty)$, we have

$$H_\alpha(C_G) \leq K_\alpha(\mathcal{S}) + \lim_{m\to\infty} F(\alpha, m)$$

$$= K_\alpha(\mathcal{S}) + \frac{1}{\alpha - 1} \log\left[\max\left(0, \frac{2\alpha - 1}{\alpha}\right)\right].$$

A. K. Yadav, and Y. Y. Shkel, "Approximation Guarantees for Minimum Rényi Entropy Functional Representations", in IEEE International Symposium on Information Theory (ISIT), 2025.

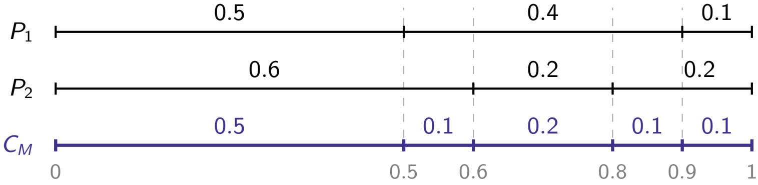
Comonotone Coupling Algorithm

- **Input :** m PMFs $\{P_i\}_{i=1}^m$, each with $\leq n$ states
- **Output :** Coupling $C_M := (c_1, c_2, \ldots, c_W)$
- **Algorithm :**
  - Sort each PMF in the non-increasing order
  - Compute the cumulative breakpoints of each PMF i.e.,
    $\mathcal{B}_i = \left\{ \sum_{j=1}^{k} P_i(j) \,:\, k \in [n-1] \right\}$
  - Merge and sort the distinct breakpoints:
    $\mathcal{B} = \{0, 1\} \cup \bigcup_{i=1}^{m} \mathcal{B}_i \quad$ i.e., $\quad 0 = b_0 < b_1 < \cdots < b_W = 1$
  - Set $c_t = b_t - b_{t-1} \;\; ; \;\; \forall\, t \in [W]$
  - Sort $(c_1, \ldots, c_W)$ in the non-increasing order

A. K. Yadav, and Y. Y. Shkel, "Couplings and Entropic Inequalities on the Majorization Lattice," 2026. (In part, accepted at IEEE Information Theory Workshop (ITW), 2026.)

Comonotone Coupling Algorithm : Example

- **Input :** $\{P_1 = (0.5, 0.4, 0.1);\ P_2 = (0.6, 0.2, 0.2)\};\ \ (m = 2\,,\ n = 3)$



- $\mathcal{B}_1 = \{0.5,\ 0.9\};\ \ \mathcal{B}_2 = \{0.6,\ 0.8\} \implies \mathcal{B} = \{0,\ 0.5,\ 0.6,\ 0.8,\ 0.9,\ 1\}$

- **Output :** Coupling $C_M = (0.5,\ 0.2,\ 0.1,\ 0.1,\ 0.1)$

Comonotone Coupling Algorithm : Worst-case Guarantee

Theorem : Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions and let C_M denote the output of the comonotone coupling algorithm. Then, for every $\alpha \in [0, \infty)$, we have

$$H_\alpha(C_M) \ \leq \ H_\alpha\Big(\bigwedge_{i=1}^m P_i \Big) + \log_2(m)$$

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Proof idea :

$$\Big(\bigwedge_{i=1}^{m} P_i \Big) \otimes \Big(\tfrac{1}{m}, \ldots, \tfrac{1}{m} \Big) \;\preceq_m\; C_M \;\preceq_m\; \bigwedge_{i=1}^{m} P_i$$

Schur concavity of H_α; the uniform factor contributes the $\log_2(m)$.

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    For Shannon entropy: additive gap ≈ 0.53 ($m = 2$)
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    For Shannon entropy: additive gap ≈ 1 ($m = 2$)
  - Neither algorithm instance-wise dominates the other
  - Both the algorithms are optimal for min-entropy i.e., $\alpha \to \infty$.

Outline

Applications : Functional Representation Lemma (FRL)

- Given jointly distributed discrete random variables (X, Y).
- **FRL:** There exists a random variable Z such that

$$X = g(Y, Z) \quad \text{and} \quad Y \perp Z$$

- Applications in network information theory, privacy, and causal inference.

A. El Gamal and Y. H. Kim, "Network Information Theory", Cambridge University Press, 2011.

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- Solving for $\min H_\alpha(Z)$ is equivalent to solving M-REC on $\{P_{X|Y=y_i}\}_{i=1}^m$.
- An Optimal Z can be chosen to be isomorphic to a minimum Rényi entropy coupling on $\{P_{X|Y=y_i}\}_{i=1}^m$.

Applications : Causal Inference

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- **Goal:** Identify the direction of causation i.e., $X \to Y$ or $Y \to X$?
- Entropy based approach to causal identifiability [Kocaoglu et al. '17]

$X \to Y$

Find exogenous random variable E s.t. $X \perp E$ and $Y = f(X, E)$ with minimum $H_\alpha(E)$.

$Y \to X$

Find exogenous random variable $\tilde{E}$ s.t. $Y \perp \tilde{E}$ and $X = g(Y, \tilde{E})$ with minimum $H_\alpha(\tilde{E})$.

Murat Kocaoglu, Alexandros G. Dimakis, Sriram Vishwanath, and Babak Hassibi, "Entropic causal inference", In Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence (AAAI), 2017.

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$X \to Y$	$Y \to X$
Find exogenous random variable E s.t. $X \perp E$ and $Y = f(X, E)$ with minimum $H_\alpha(E)$.	Find exogenous random variable $\tilde{E}$ s.t. $Y \perp \tilde{E}$ and $X = g(Y, \tilde{E})$ with minimum $H_\alpha(\tilde{E})$.

- $X \to Y$ if $H_\alpha(X) + H_\alpha(E^*) < H_\alpha(Y) + H_\alpha(\tilde{E}^*)$ and vice-versa.

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Applications : Multi-Modal Learning

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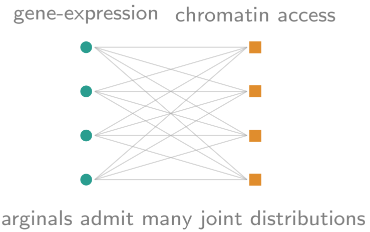
- Each modality is observed only through its **marginal** — which joint distribution should we learn?



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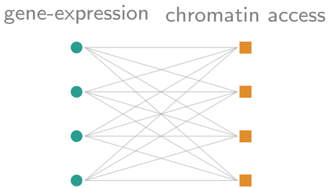
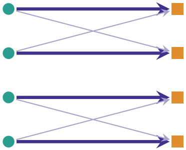


marginals admit many joint distributions



minimum entropy coupling — the most informative dependence

- Continuous MEC via two cooperative conditional **diffusion models** (DDMEC).

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Other Applications

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Thank You!