Smarter Decisions for a Better World

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Minimum Rényi Entropy Couplings (and Applications)

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Minimum Rényi Entropy Coupling (M-REC)

Given : m marginal distributions $\{P_1, P_2, \ldots, P_m\}$

Find : coupling $C(X_1, \ldots, X_m)$.....

...with minimum $H_\alpha(C)$ $(\forall \alpha \geq 0)$

Such that : $X_i \sim P_i$; $\forall i \in [m]$.

Applications : Causal Inference

- Given jointly distributed discrete random variables (X, Y).
- **Goal:** Identify the direction of causation i.e., $X \to Y$ or $Y \to X$?
- Entropy based approach to Causal Identifiability [Kocaoglu et al. '17]

$X \to Y$	$Y \to X$
Find Exogenous random variable E s.t. $X \perp E$ and $Y = f(X, E)$ with minimum $H_\alpha(E)$.	Find Exogenous random variable $\tilde{E}$ s.t. $X \perp E$ and $X = g(Y, \tilde{E})$ with minimum $H_\alpha(\tilde{E})$.

Murat Kocaoglu, Alexandros G. Dimakis, Sriram Vishwanath, and Babak Hassibi, "Entropic causal inference", *In Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence (AAAI'17),* **AAAI Press, 1156–1162.**

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$X \to Y$

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s.t.
$X \perp E$ and $Y = f(X, E)$
with minimum $H_\alpha(E)$.

$Y \to X$

Find Exogenous random variable $\tilde{E}$
s.t.
$X \perp E$ and $X = g(Y, \tilde{E})$
with minimum $H_\alpha(\tilde{E})$.

- $X \to Y$ if $H_\alpha(X) + H_\alpha(E) \leq H_\alpha(Y) + H_\alpha(\tilde{E})$ and vice-versa.
- Computing E with minimum $H_\alpha(E)$ $\equiv$ solving M-REC problem on $\{P_{Y|X=x}\}_{x \in \mathcal{X}}$.

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Applications : Secrecy by Design

- Private database $\mathbf{X} := (X_1,\ X_2,\ \ldots,\ X_n)$ to be used for some statistical task.
- Efficiently release the sanitized version of the database i.e., $\mathbf{Z} := (Z_1,\ Z_2,\ \ldots,\ Z_n)$.
- **Naive approach :** ensure 'perfect secrecy' i.e., $I(X_i\ ;\ Z_i) = 0\ ;\ \forall\ i \in [n]$.

Y. Y. Shkel, R. S. Blum and H. V. Poor, "Secrecy by Design With Applications to Privacy and Compression," in *IEEE Transactions on Information Theory*, vol. 67, no. 2, pp. 824-843, Feb. 2021.

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- Relaxing perfect secrecy : **Secrecy by Design**
- Identify the sensitive information in $\mathbf{X}$ (function of $\mathbf{X}$) i.e., $\mathbf{S} := (S_1,\ S_2,\ \ldots,\ S_n)$.
- Release $\mathbf{Z}$ ensuring the 'perfect secrecy' of $\mathbf{S}$ i.e.,

$$I(S_i\ ;\ Z_i) = 0\ ;\ \forall\ i \in [n] \quad \text{and} \quad X_i = f_i(S_i, Z_i)\ ;\ \forall\ i \in [n]$$

and that the entropy of Z_i ; $\forall\ i \in [n]$, is minimum.

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and that the entropy of Z_i ; $\forall\ i \in [n]$, is minimum.

- Equivalent to solving M-REC problem on $\{P_{X_i|S_i=s}\}_{s\in\mathcal{S}_i}$ for every $i \in [n]$.

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Other Applications…

- Perfectly Secure Steganography
- Random Number Generation
- Dimensionality Reduction
- Network Information Theory (FRL)
- Contingency Tables
- Transportation Polytopes ……….

F. Cicalese, L. Gargano and U. Vaccaro, "Minimum-Entropy Couplings and Their Applications," in *IEEE Transactions on Information Theory*, vol. 65, no. 6, pp. 3436-3451, June 2019.

However …

- Computing $H_\alpha(C^\star)$ is a **NP-hard** problem in the support size of PMFs.

- **Lower bounds on** $H_\alpha(C^\star)$ — Converse type results
- **Upper bounds on** $H_\alpha(C^\star)$ — Achievability type results

Lower bounds (Converse Results)

What are the worst-case guarantees on $H_\alpha(C^\star)$?

*Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations ," in *IEEE International Symposium on Information Theory (ISIT)*, 2023.

Prelude

Let X be a random variable such that $X \sim P_X$:

Information of X :

$$\imath_X(x) := \log\left(\frac{1}{P_X(x)}\right) \; ; \text{w. p. } P_X(x).$$

Information spectrum of X :

$$\mathbb{F}_{\imath_X(t)} = \mathbb{P}[\imath_X(X) \leq t] \quad ; \forall t \in [0,\infty)$$

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Shannon entropy of X :

$$H(X) = \mathbb{E}[\imath_X(X)]$$

$$= \int_0^\infty \left(1 - \mathbb{F}_{\imath_X}(t)\right) dt$$

Rényi entropy of X :

$$H_\alpha(X) = \frac{1}{1-\alpha} \log\left(\mathbb{E}[2^{(1-\alpha)\imath_X(X)}]\right) ;$$

$$\forall \alpha \in [0,\infty)$$

Majorization ($\preceq_m$)

Definition

given $Q = (q_1, q_2, q_3, \ldots,) \,; \quad q_1 \geq q_2 \geq \ldots$

$P = (p_1, p_2, p_3, \ldots,) \,; \quad p_1 \geq p_2 \geq \ldots$

we say $Q \preceq_m P$

if $$\sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i, \quad \forall k = 1,2,\ldots$$

$\preceq_m$ forms a partial order and a complete lattice.

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Greatest Lower Bound of $\{P_1, P_2, \ldots, P_m\}$

$$\bigwedge_{i=1}^{m} P_i \preceq_m P_i \quad ; \quad \forall i \in [m]$$

$$Q \preceq_m P_i \;; \forall i \in [m] \implies \bigwedge_{i=1}^{m} P_i \preceq_m P_i$$

Schur Concavity

$$Q \preceq_m P \implies H_\alpha(Q) \geq H_\alpha(P)$$

Lower bound : A very basic one

- $C(X_1, \cdots, X_m) \sqsubseteq P_i$; $\forall C,\ \forall i \in [m]$
- Aggregation implies Majorization.
- $C(X_1, \cdots, X_m) \preceq_m P_i$; $\forall C,\ \forall i \in [m]$

$$H_\alpha(C^\star) \geq \max_{i \in [m]} H_\alpha(X_i)$$

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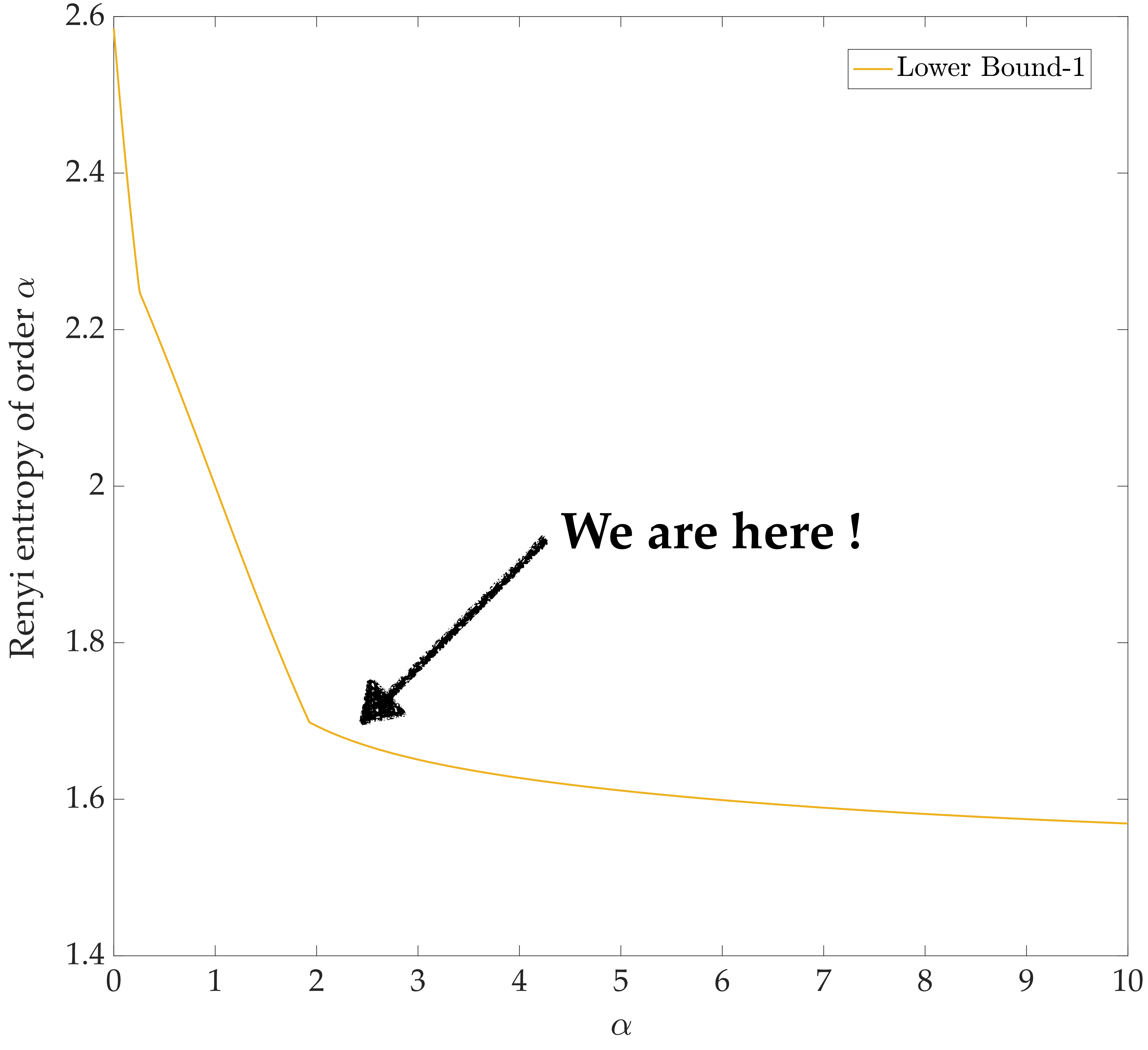
Toy Example with $(m = 3, n = 6)$:

$P_1 = (0.5, 0.125, 0.125, 0.125, 0.125, 0)$;

$P_2 = (0.4, 0.4, 0.1, 0.1, 0, 0)$;

$P_3 = (0.35, 0.35, 0.25, 0.04, 0.005, 0.005)$;



Lower bound : Based on Majorization ($\preceq_m$)

Recall,

$$C(X_1, \cdots, X_m) \preceq_m P_i \ ; \forall C, \forall i \in [m]$$

Thus,

$$C(X_1, \cdots, X_m) \ \preceq_m \bigwedge_{i=1}^{m} P_i \ \preceq_m P_i \ ; \forall i \in [m]$$

$$\boxed{H_\alpha(C^\star) \geq H_\alpha\Big(\bigwedge_{i=1}^{m} P_i\Big)}$$

F. Cicalese, L. Gargano, and U. Vaccaro, "Minimum-entropy couplings and their applications," ***IEEE Transactions on Information Theory*****, vol. 65, no. 6, pp. 3436–3451, 2019.**

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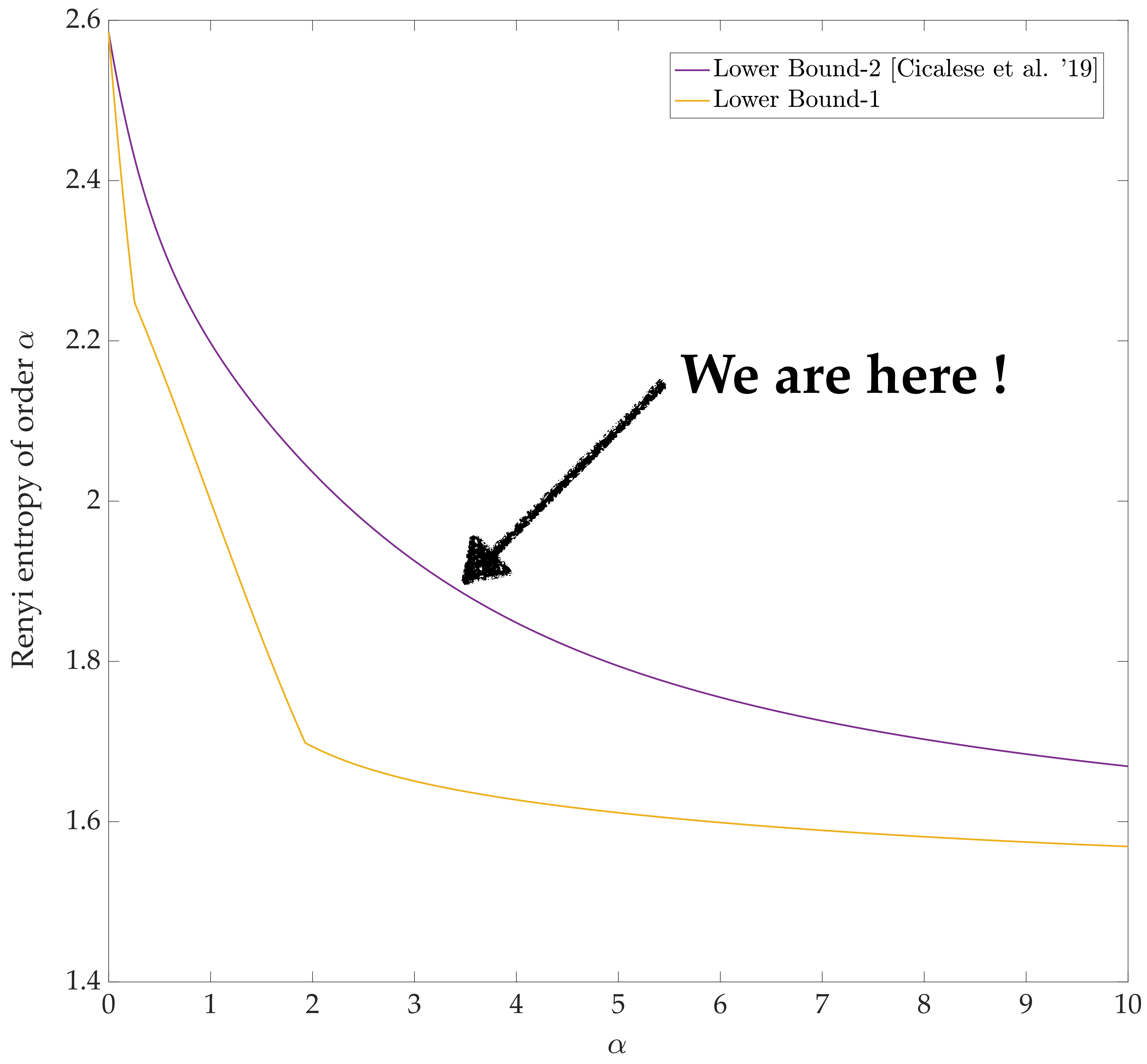

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Majorization : Information-spectrum sense ($\preceq_\imath$)

We say : P majorizes Q in an information-spectrum sense, i.e.,

$$Q \preceq_\imath P$$

if

$$\mathbb{F}_{\imath_Q}(t) \leq \mathbb{F}_{\imath_P}(t), \quad \forall t \in [0,\infty)$$

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Lemma 1 : $Q \preceq_\imath P \implies Q \preceq_m P$

Information- spectrum based Lower bound

Main Result I

Theorem : Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions with support size atmost n. Then,

$$C \preceq_\iota P_i \;\; ; \;\; \forall C, \;\forall i \in [m]$$

or

$$\mathbb{F}_{\iota_C}(t) \leq \mathbb{F}_{\iota_{X_i}}(t) \; ; \;\; \forall t \in [0,\infty) \;, \forall C, \;\forall i \in [m]$$

Information- spectrum based Lower bound

Main Result I

$$\mathbb{F}_{\imath_{C^\star}}(t) \leq \mathbb{F}_{\imath_{X_i}}(t)\ ; \quad \forall i \in [m]$$

$$\Downarrow$$

$$H(C^\star) = \mathbb{E}[\imath_{C^\star}(C^\star)] = \int_0^\infty \left(1 - \mathbb{F}_{\imath_{C^\star}}(t)\right)dt$$

$$\geq \int_0^\infty \max_{i\in[m]} \left(1 - \mathbb{F}_{\imath_{X_i}}(t)\right)dt$$

$$\boxed{H(C^\star) \geq K(\mathcal{S})} \quad ;\ \text{where } K(\mathcal{S}) := \int_0^\infty \max_{i\in[m]} \left(1 - \mathbb{F}_{\imath_{X_i}}(t)\right)dt$$

Similarly extended for Rényi Entropy i.e.,
$H_\alpha(C^\star) \geq K_\alpha(\mathcal{S})$.

Information- spectrum based Lower bound

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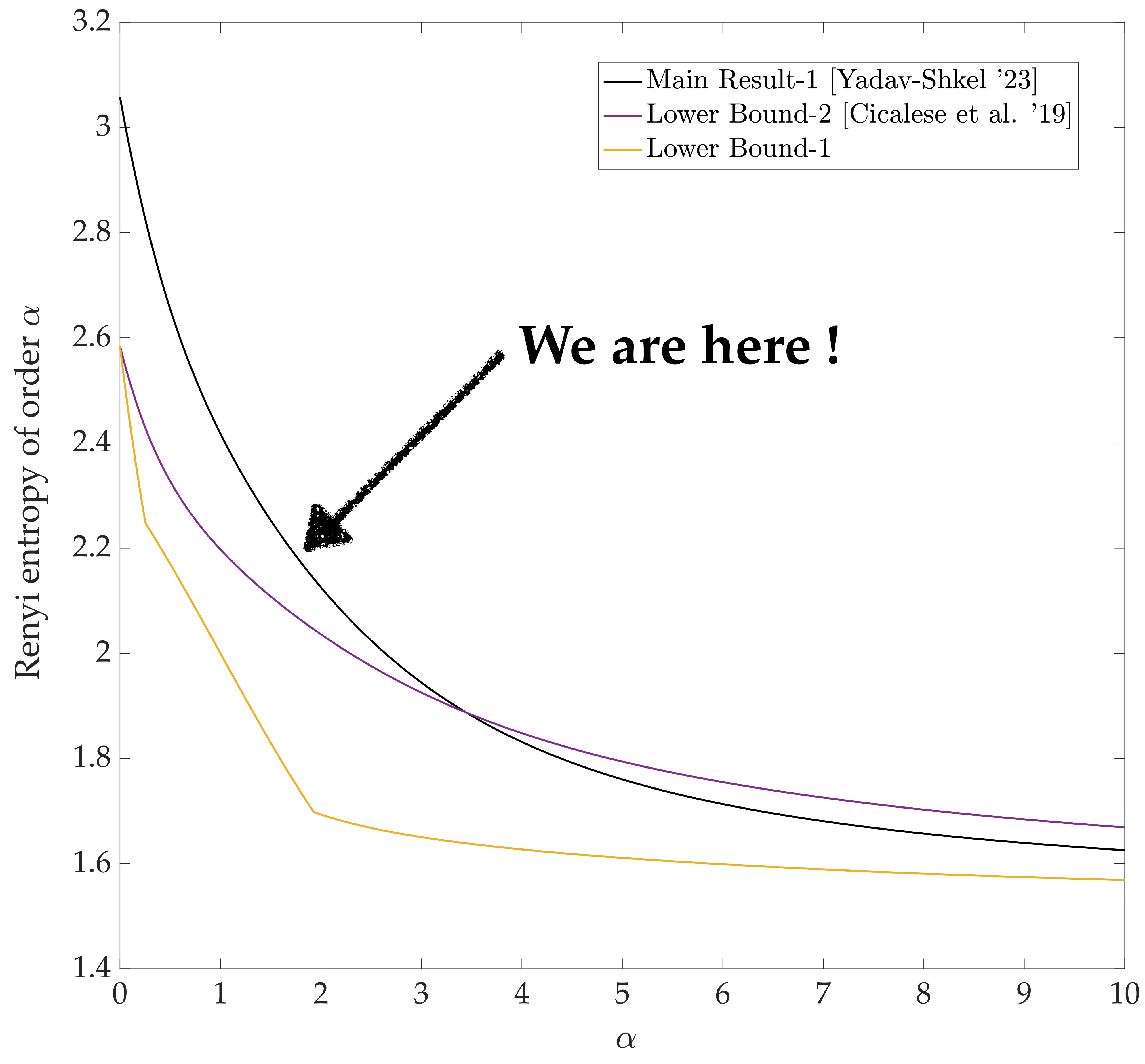

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$$Q \preceq_\imath P$$

$$\text{if} \quad \mathbb{F}_{\imath_Q}(t) \leq \mathbb{F}_{\imath_P}(t), \quad \forall t \in [0,\infty)$$

Recall that :

$$Q \preceq_m P$$

$$\text{if} \quad \sum_{i=1}^{k} q_i \leq \sum_{i=1}^{k} p_i \quad \forall k \in [m]$$

- **Lemma 1:** $Q \preceq_\imath P \implies Q \preceq_m P$
- Let $\mathscr{F} = \{Q : Q \preceq_\imath P_i \, ; \; \forall i \in [m]\}$

$$\nexists \bigwedge_{i=1}^{m} P_i \in \mathscr{F} \quad \text{s.t.} \quad Q \preceq_\imath \bigwedge_{i=1}^{m} P_i \preceq_\imath P_i \, ; \; \forall i \in [m] \, , \; Q \in \mathscr{F}$$

$\preceq_\imath$ doesnot form a lattice ; the greatest lower bound doesnot exist.

Majorization : Information-spectrum sense ($\preceq_\imath$)

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- **Lemma 1:** $Q \preceq_\imath P \implies Q \preceq_m P$
- **Lemma 2:** Let $\mathcal{F} = \{Q : Q \preceq_\imath P_i \quad \forall i \in [m]\}$

$$\exists Q^* \in \mathcal{F} \text{ s.t. } Q \preceq_m Q^* ; \quad \forall Q \in \mathcal{F}$$

Information- spectrum based Lower bound

Main Result II

Recall : $C \preceq_\iota P_i \;\; ; \qquad \forall C, \; \forall i \in [m]$

Define : $\mathcal{S} = \{Q : \; Q \preceq_\iota P_i \,; \; i \in [m]\}$

$\forall C$, we have that $C \in \mathcal{S}$. Furthermore, from Lemma 2, we have:

$$\exists Q^* \in \mathcal{S} \text{ s.t. } Q \preceq_m Q^* \preceq_i P_i \quad ; \quad \forall Q \in \mathcal{S}$$

Therefore,

$$C^\star \preceq_m Q^* \preceq_\iota P_i \,; \; \forall i \in [m].$$

$$\boxed{H_\alpha(Z) \geq H_\alpha(Q^*)}$$

Information- spectrum based Lower bound

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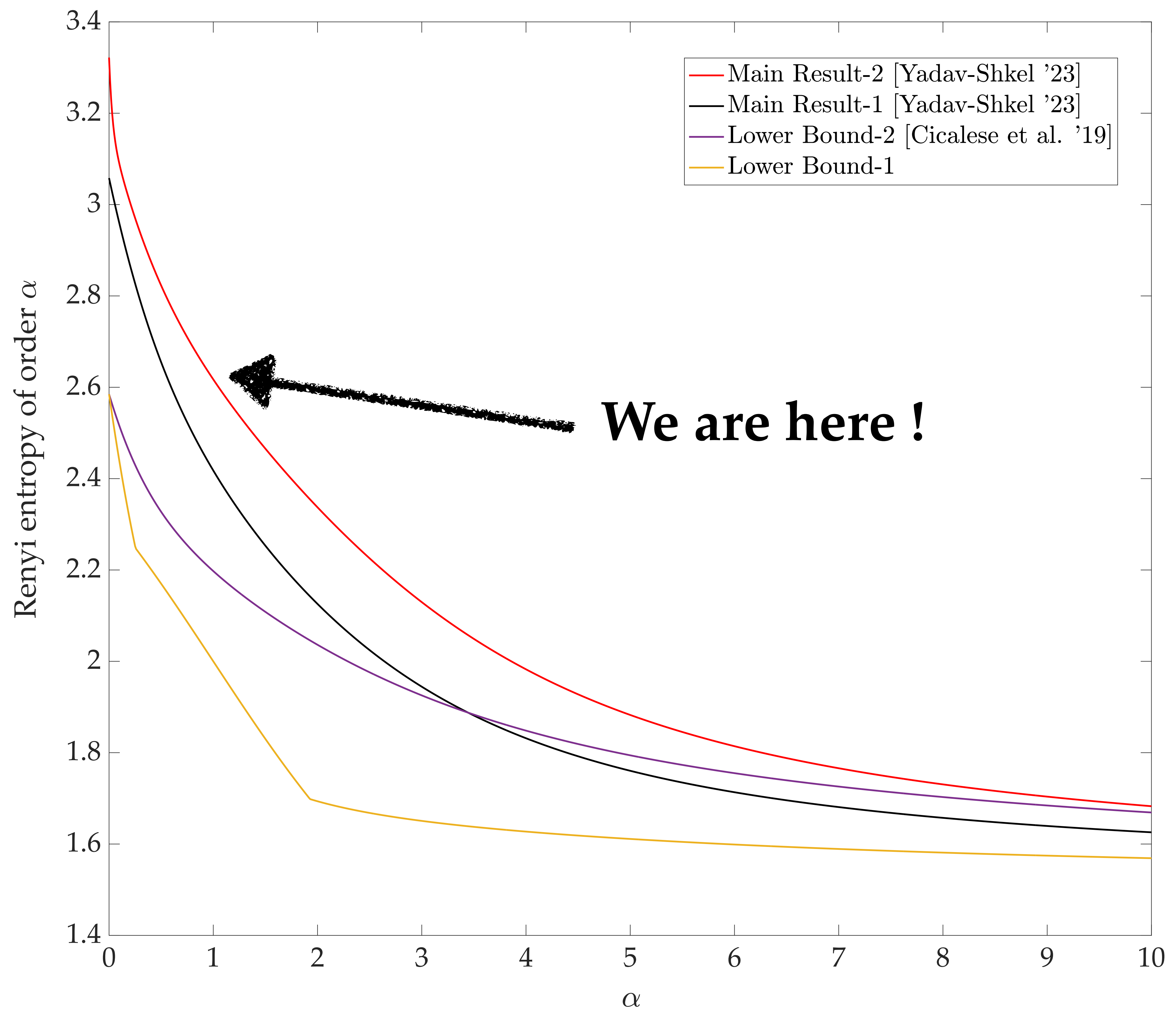

Upper bounds (Achievability Results)

Can we construct ‘nice’ couplings and give some approximation guarantees w.r.t $H_\alpha(C^\star)$?

A. K. Yadav, and Y. Y. Shkel, “Approximation Guarantees for Minimum Rényi Entropy Functional Representations” , in *IEEE International Symposium on Information Theory (ISIT)*, 2025.

Information-spectrum based Lower Bound : Main Result 1

Theorem : Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions. Then, for any $\alpha \in [0,\infty)$, we have

$$H_\alpha(C^\star) \geq K_\alpha(\mathcal{S})$$

where, $$K_\alpha(\mathcal{S}) = \begin{cases} \frac{1}{1-\alpha} \log \left[1 + \int_0^\infty J_\alpha(t)dt\right] & ; \text{if } \alpha \in [0,1) \cup (1,\infty) \\ \int_0^\infty G(t)dt & ; \alpha = 1 \end{cases}$$

such that : $$G(t) := \max_{i \in [m]} \left(1 - \mathbb{F}_{\imath_{X_i}}(t)\right)$$

$$J_\alpha(t) := (\ln 2)(1-\alpha)G(t)2^{(1-\alpha)t}$$

***Y. Y. Shkel, and *A. K. Yadav, "Information-spectrum converse for minimum entropy couplings and functional representations ," in *IEEE International Symposium on Information Theory (ISIT)*, 2023.**

Upper Bounds

- Approximation analysis based on the Greedy Coupling Algorithm [Kocaoglu et al.]
  - Let C_Z denote the output of the algorithm
  - $K_\alpha(\mathcal{S}) \leq H_\alpha(C^\star)$ - - [from the Lower bound]
  - $K_\alpha(\mathcal{S}) \leq H_\alpha(C^\star) \leq H_\alpha(C_Z)$ - - [problem's nature]
  - **Our Goal :** $H_\alpha(C_Z) \leq K_\alpha(\mathcal{S}) + Q$; (finding the smallest Q for every $\alpha \in [0,\infty)$).

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Greedy Coupling Algorithm

- **Input :** m PMFs $\{P_i\}_{i=1}^m$, each with $\leq n$ states
- **Output :** Coupling $C_Z := (c_1, c_2, \ldots, c_T)$

- Sort each PMF in the non-increasing order
- Find the minimum of maximum of each PMF i.e., $q = \min_i (P_i(1))$

- Append q as the next state of C_Z
- Update the maximum state of each PMF
- i.e., $P_i(1) = \left(P_i(1) - q\right)$, $\forall\, i \leq m$
- Sort each PMF in non-increasing order
- Find $q = \min_i (P_i(1))$

} **Repeat until** $q > 0$

Greedy Coupling Algorithm : Example

- **Input :** $\{P_1 = (0.5,0.4,0.1)\,;P_2 = (0.6,0.2,0.2)\}$; $(m = 2\,, n = 3)$

Iteration (t)	Current PMFs	q	Updated PMFs	C_Z
1	(0.5, 0.4, 0.1) (0.6, 0.2, 0.2)	0.5	(0, 0.4, 0.1) (0.1, 0.2, 0.2)	(0.5)
2	(0.4, 0.1, 0) (0.2, 0.2, 0.1)	0.2	(0.2, 0.1, 0) (0, 0.2, 0.1)	(0.5, 0.2)
3	(0.2, 0.1, 0) (0.2, 0.1, 0)	0.2	(0, 0.1, 0) (0, 0.1, 0)	(0.5, 0.2, 0.2)
T = 4	(0.1, 0, 0) (0.1, 0, 0)	0.1	(0, 0, 0) (0, 0, 0)	(0.5, 0.2, 0.2, 0.1)
5	(0, 0, 0) (0, 0, 0)	0	(0, 0, 0) (0, 0, 0)	

- **Output :** Coupling $C_Z = (0.5,0.2,0.2,0.1)$

Main Results

Theorem : Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions of support size n. Then, for any $\alpha \in [0,\infty)$, we have

$$H_\alpha(C_Z) \leq K_\alpha(\mathcal{S}) + F(\alpha, m)$$

where, $$F(\alpha, m) = \frac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$$
$$r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m)).$$

where, $$\tilde{r}(\alpha, m) := \begin{cases} \max_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \leq w_3 \leq \ldots \leq w_m < w_{m+1}.}} \sum_{k=2}^{m} w_k (w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); & \text{for } \alpha \in [0,1), \\ \min_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \leq w_3 \leq \ldots \leq w_m < w_{m+1}.}} \sum_{k=2}^{m} w_k (w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); & \text{for } \alpha \in (1,\infty). \end{cases}$$; and $$w_k := \frac{p_k^t(1) - p_1^t(1)}{p_1^t(1)}$$

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where, $$F(\alpha, m) = \frac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$$
$$r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m)).$$

Consequently,

$$K_\alpha(\mathcal{S}) \leq H_\alpha(C^\star) \leq H_\alpha(C_Z) \leq K_\alpha(\mathcal{S}) + F(\alpha, m)$$
$$\leq H_\alpha(C^\star) + F(\alpha, m)$$

Main Results

Corollary 1 : Let $\mathcal{S} := \{P_1, P_2\}$ be the set of two marginal distributions of support size n. Then, for any $\alpha \in [0, \infty)$, we have

$$H_\alpha(C_Z) \leq K_\alpha(\mathcal{S}) + F(\alpha, 2)$$

$$\text{where,} \quad F(\alpha, 2) = \frac{1}{\alpha - 1} \log \left[1 + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha - 1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha - 1}} \right].$$

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$$H_\alpha(C_Z) \leq K_\alpha(\mathcal{S}) + F(\alpha,2)$$

where, $$F(\alpha,2) = \frac{1}{\alpha-1}\log\left[1 + \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right].$$

$F(\alpha, m)$ doesnot have a closed-form solution, in general!

Main Results

Recall that $F(\alpha, m) = \frac{1}{\alpha - 1} \log \left[r(\alpha, m) \right]$; where $r(\alpha, m) = \max(0, 1 - \tilde{r}(\alpha, m))$ such that

$$\tilde{r}(\alpha, m) := \begin{cases} \max_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}.}} \sum_{k=2}^{m} w_k (w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); \ \text{for } \alpha \in [0,1), \\ \min_{\substack{w_1 = 0; \\ w_{m+1} = 1; \\ w_1 < w_2 \le w_3 \le \ldots \le w_m < w_{m+1}.}} \sum_{k=2}^{m} w_k (w_k^{\alpha-1} - w_{k+1}^{\alpha-1}); \ \text{for } \alpha \in (1,\infty). \end{cases}$$

Lemma : For every $\alpha \in [0,\infty)$, $F(\alpha, m)$ is an non-decreasing function of m.

As $m \to \infty$, $F(\alpha, m)$ approaches $\frac{1}{\alpha - 1} \log \left[\max \left(0, \frac{2\alpha - 1}{\alpha} \right) \right]$.

Main Results

Corollary 2 : Let $\mathcal{S} := \{P_1, \ldots, P_m\}$ be the set of m marginal distributions of support size n. Then, for any $\alpha \in [0,\infty)$, we have

$$H_\alpha(C_Z) \leq K_\alpha(\mathcal{S}) + \lim_{m\to\infty} F(\alpha, m)$$

$$= K_\alpha(\mathcal{S}) + \frac{1}{\alpha - 1} \log \left[\max\left(0, \frac{2\alpha - 1}{\alpha}\right) \right].$$

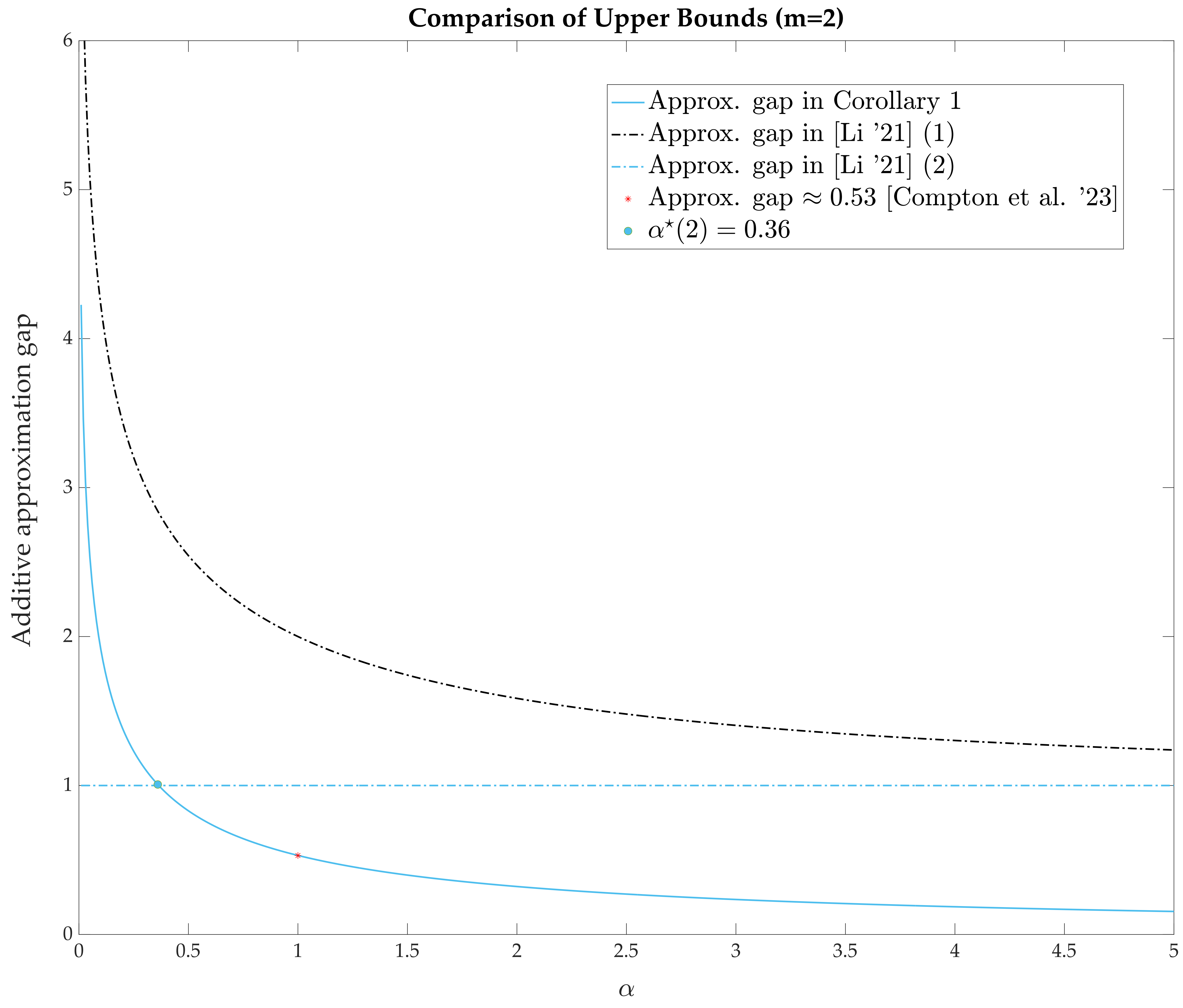
Comparison of Upper Bounds : ($m = 2$)

[Li, Trans. IT '21] (1) : $H_\alpha(\tilde{Z}) \leq H_\alpha(C^\star) + \begin{cases} \infty & ;\ \textbf{if } \alpha = 0 \\ 2 & ;\ \textbf{if } \alpha = 1 \\ 1 & ;\ \textbf{if } \alpha = \infty \\ \frac{-\alpha - \log(1 - 2^{-\alpha})}{1 - \alpha} & ;\ \textbf{otherwise} \end{cases}$

[Li, Trans. IT '21] (2) : $H_\alpha(\tilde{Z}) \leq H_\alpha(C^\star) + 1$.

[Our Work] : $H_\alpha(C_Z) \leq H_\alpha(C^\star) + F(\alpha, 2)$.

Comparison of Upper Bounds : ($m = 2$)

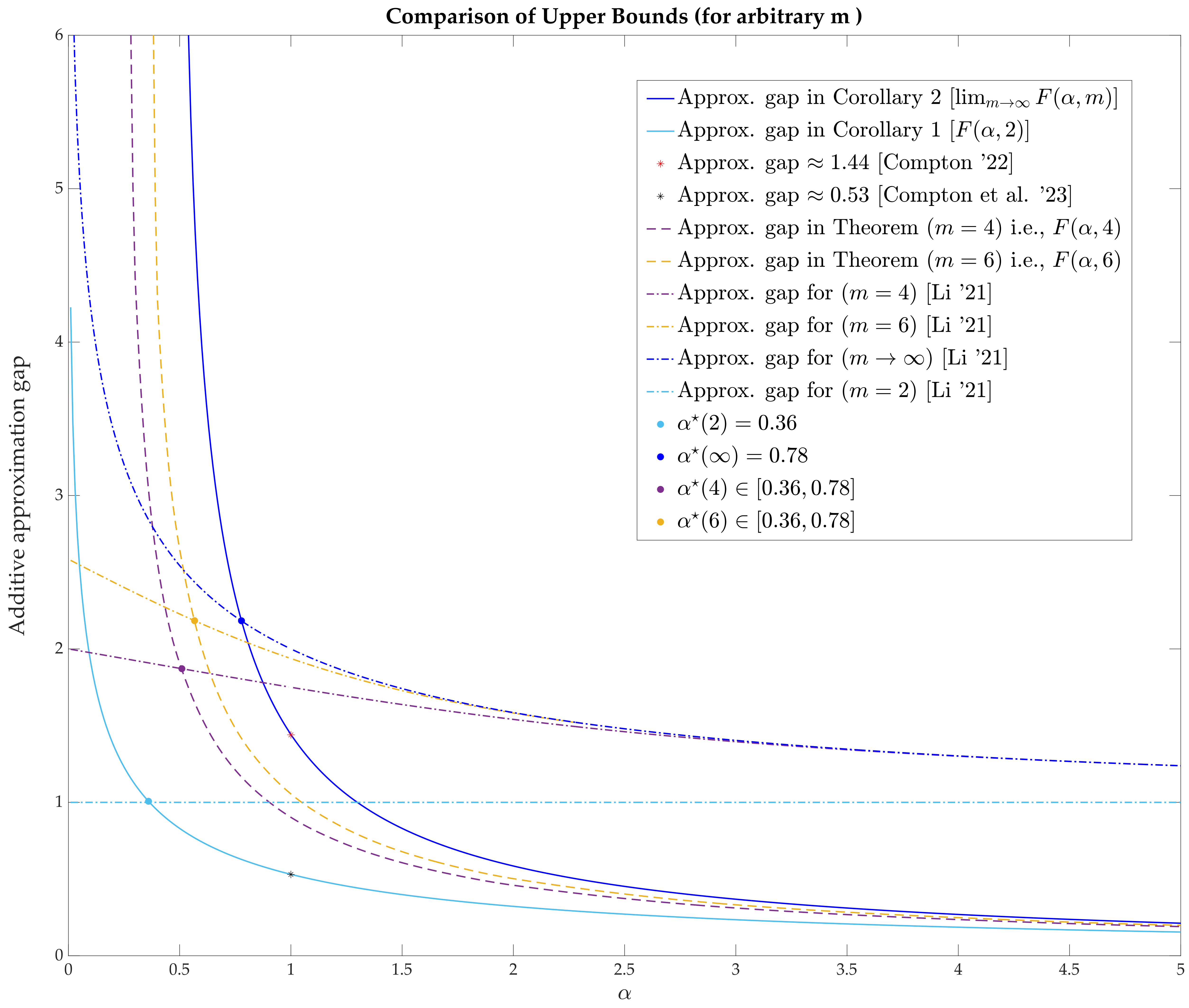


Comparison of Upper Bounds : (arbitrary m)

[Li, Trans. IT ′21] (2) : $H_\alpha(\tilde{Z}) \leq H_\alpha(C^\star) + \frac{1}{1-\alpha} \log \left(\frac{(2^\alpha - 2)2^{-\alpha m} + 2^{-\alpha}}{1 - 2^{-\alpha}} \right)$

[Our Work] : $H_\alpha(C_Z) \leq H_\alpha(C^\star) + F(\alpha, m) \leq H_\alpha(C^\star) + \frac{1}{\alpha - 1} \log \left[\max\left(0, \frac{2\alpha - 1}{\alpha}\right) \right].$

Comparison of Upper Bounds : (arbitrary m)



Summary

- **Converse type results (Lower Bounds) :**
  - Two lower bounds based on 'Information-spectrum majorization'.
  - $K_\alpha(\mathcal{S})$ is better for lower values of α.
  - Q^* is better than all the previously known lower bounds for any $\alpha \in [0,\infty)$.

- **Achievability type results (Upper Bounds) :**
  - Approximation analysis between the Rényi entropy of the 'output of the greedy coupling algorithm' and the 'optimal coupling'.
  - Our analysis is better for high values of α i.e., $\alpha \geq a^\star(m)$,
    where $\alpha^\star(m) \in [0.36,0.78]$ for every $m \geq 2$.
  - Greedy Coupling Algorithm is optimal for min-entropy i.e., $\alpha \to \infty$.